Monge-optimal martingale couplings

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A canonical optimal martingale coupling problem

Let $\mathcal{M}(\mu, \nu) = \{(X, Y) : X \sim \mu, Y \sim \nu, \; \mathbb{E}[Y|X] = X\}$.

The primal problem:

$$\inf_{(X,Y)\in\mathcal{M}(\mu,\nu)} \mathbb{E}[|Y - X|].$$

The Lagrangian approach

$$\min_{\rho}\left\{\int\int|y-x|\rho(dx,dy)\right\} \text{ subject to}$$

$$\int_x \rho(dx,dy)=\nu(dy),\ \int_y \rho(dx,dy)=\mu(dx),$$

$$\int_y (y-x)\rho(dx,dy)=0.$$

- $L(x,y)=|y-x|-\alpha(y)-\beta(x)-\theta(x)(x-y).$
- $\int\int L(x,y)\rho(dx,dy)+\int\alpha(y)\nu(dy)+\int\beta(x)\nu(dx)$

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Classical economic motivation

Let $\mathcal{S}$ be set of triples (α, β, θ) such that $L(x, y) \geq 0$ where $L(x, y) = |y - x| - \alpha(y) - \beta(x) - \theta(x)(x - y)$.
For such 'triples' in $\mathcal{S}$,

$$\mathbb{E}[|Y - X|] \geq \int \beta(x)\mu(dx) + \int \alpha(y)\nu(dy)$$

$(F_t)_{T_0 \leq t \leq T_1}$ forward price process (a martingale).
Suppose marginal laws $F_{T_0} \sim \mu$ and $F_{T_1} \sim \nu$ are known.

- Purchase $\beta(F_{T_0})$ and $\alpha(F_{T_1})$
- Sell forward $\theta(F_{T_0})$ of stock over $[T_0, T_1]$
- Payoff is $\alpha(F_{T_1}) + \beta(F_{T_0}) - \theta(F_{T_0})(F_{T_1} - F_{T_0}) \leq |F_{T_1} - F_{T_0}|$
- The strategy is a *sub-hedge*.

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Plan:

1.) Re-visit classical optimal transport ('gradient principle')
2.) A structural link to Economics: Mechanism design
3.) Solve the canonical m-g coupling problem.

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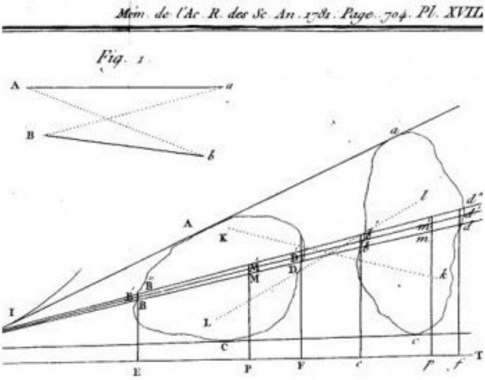

- $c(x, y) = |x - y|$: No splitting, transport along straight lines, 'forbidden' paths
- More on Monge: Hardy Lecture '12, Étienne Ghys.

http://www.dailymotion.com/video/xri60u_ems-130-ghys_tech

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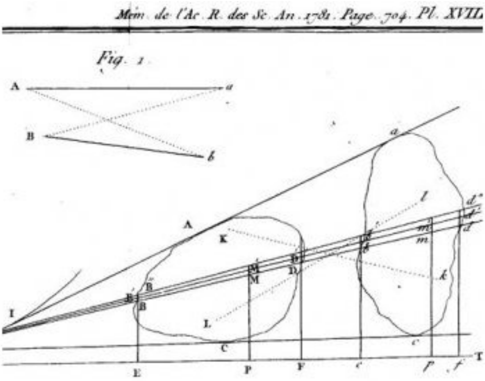

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The Monge transport problem

Monge's 'earthwork' problem in one dimension:

Transport earth from a given area to a target site in a way that minimises the cost of carriage.

$$\inf_{s : s_* \mu = \nu} \int_{\mathbb{R}} |x - s(x)| \mu(dx).$$

The $s_* \mu$ are push-forward maps of μ through s.
$s_* \mu(U) = \mu(s^{-1}(U))$ for Borel sets $U \subseteq \mathbb{R}$.

The Kantorovich relaxation and duality

Let $\mathcal{C}(\mu, \nu) = \{(X, Y) : X \sim \mu, Y \sim \nu\}$.

The Kantorovich relaxation (primal problem):

$$\inf_{(X,Y)\in\mathcal{C}(\mu,\nu)} \mathbb{E}[c(X, Y)].$$

The dual problem:

$$\sup_{\Psi,\Phi} \left\{ \int_{\mathbb{R}} \Psi(x)\mu(dx) + \int_{\mathbb{R}} \Phi(y)\nu(dy) \ ; \ \Psi(x) + \Phi(y) \leq c(x, y) \right\}.$$

Technology: c-convexity and Rüschendorf's theorem

(X, Y) is *c-optimal* if
$\mathbb{E}[c(X,Y)] = \sup\{\mathbb{E}[c(U,V)]; U \sim \mu, V \sim \nu\}$.

c-conjugate	$f^*(y)$	$=$	$\sup_x[c(x,y) - f(x)]$
c-subgradient	$\partial^c f(x)$	$=$	$\{z : f(x) + f^*(z) = c(x,z)\}$
c-convex	$(f^*)^*$	$=$	f

Theorem: (Rüschendorf '91)
If $c(x,y)$ is lower majorized ($c(x,y) \geq p(x) + q(y)$, $p, q \in L^1$) and
$\inf_{\Psi,\Phi}\left\{\int_{\mathbb{R}} \Psi(x)\mu(dx) + \int_{\mathbb{R}} \Phi(y)\nu(dy);\ \Psi(x) + \Phi(y) \geq c(x,y)\right\} < \infty$ then
(X, Y) is a c-optimal pair if and only if $Y \in \partial^c\Psi(X)$ for some c-convex Ψ.

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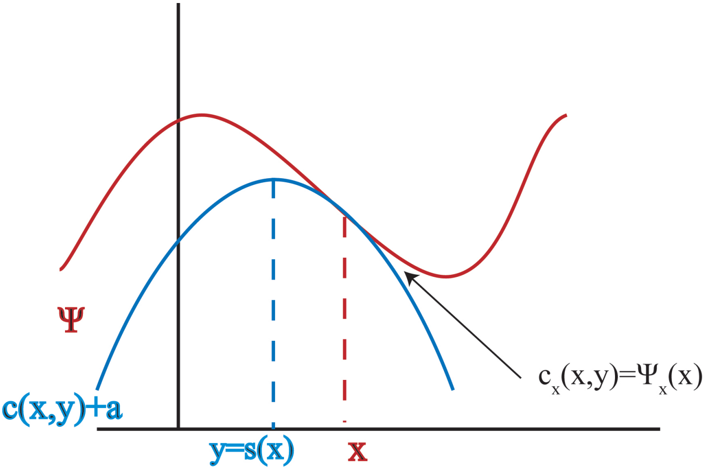
Illustration: The gradient principle



Monotone structure

c is *supermodular* if:

$c(x, y) - c(x', y)$ is increasing in y for all $x > x'$.

(if c is differentiable, $\frac{\partial^2}{\partial x \partial y} c(x, y) \geq 0$.)

Monotonicity Theorem (Topkis):
F supermodular $\Rightarrow$ $argmax_z\{F(x, z)\}$ non-decreasing.

Monotone transportation plans:

$$\sup\{\mathbb{E}[c(U, V)]; U \sim \mu, V \sim \nu\}.$$

i.) c supermod. $\Rightarrow (X, Y) \in (X, \partial^c \Psi(X))$, $\partial^c \Psi$ increasing.

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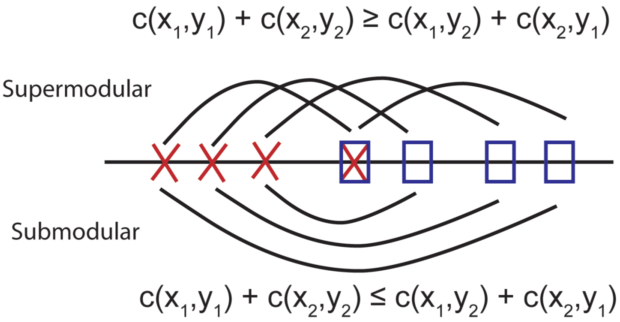

Gangbo/McCann: $\inf\{\mathbb{E}[c(U,V)]; U \sim \mu, V \sim \nu\}$.

i.) $c(x,y) = h(y-x)$, h strict. convex $\Rightarrow$ increasing structure.

ii.) $c(x,y) = h(|y-x|)$, h strict. concave $\Rightarrow$ decr. structure.

An'economy of scale for longer trips'.

Monge type Principles: No crossing, stay if you can.

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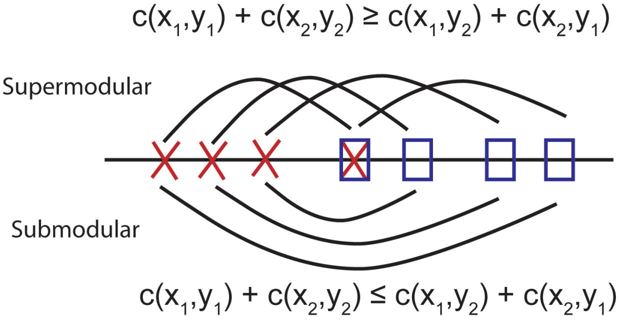

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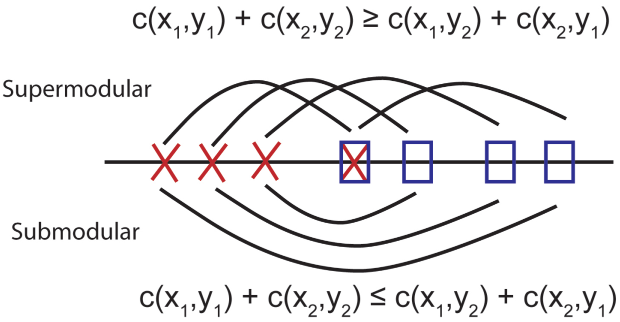

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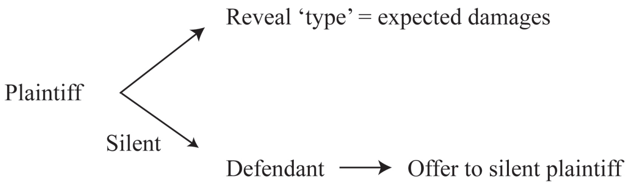
Example: Supermodular mechanism

Pretrial negotiations: Shavell



Plaintiff’s threshold problem:	When to stay silent?
Defendant’s problem:	Offer to a silent plaintiff?

Supermodular structure:

Plaintiff:	Stay silent if offers to silent plaintiffs are high.
Defendant:	Make high initial offers when plaintiffs adopt a high strategy (high offer will stop a lawsuit).

Optimal transport $\leftrightarrow$ Mechanism design

- Task: Design 'sensible' financial contracts (investment mechanisms)

- For instance: What is the best way of defining variance?

  Two immediate choices:

i.) $\sum_i (\log(f_{t_i}) - \log(f_{t_{i-1}}))^2$ or as

ii.) $\sum_i \left(\frac{f_{t_i} - f_{t_{i-1}}}{f_{t_{i-1}}} \right)^2$

iii.) Other options: Bondarenko kernel, Martin's simple kernel, etc.

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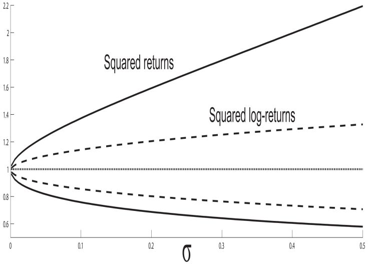
Mechanism design: Variance swap

Kernel	Structure	Embedding	Ex. Model
$\left(\frac{y-x}{x}\right)^2$	increasing	A-Y	drift + single jump
$(\log(y)-\log(x))^2$	decreasing	Perkins	drift+no jump /large jump



$$T=1/12,\ X_\sigma=e^{\sigma N-\sigma^2/2},\ \text{VIX}=\mathbb{E}[-2\log X_{\sigma/\sqrt{12}}]$$

The canonical optimal martingale coupling problem

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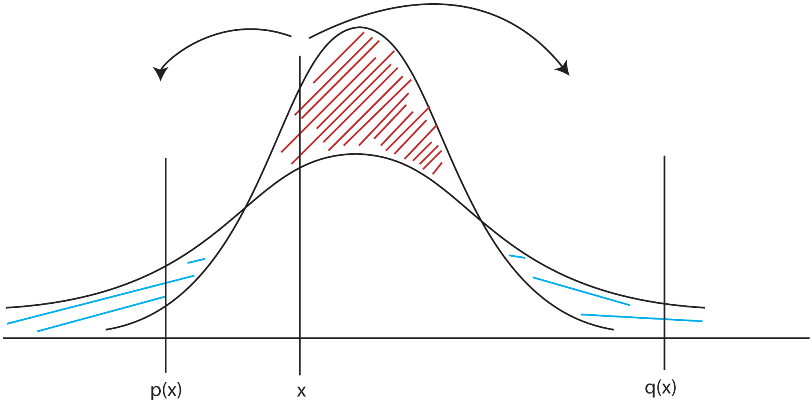
- *Primal problem:*

$$\inf_{(X,Y)\in\mathcal{M}(\mu,\nu)} \mathbb{E}[|Y - X|].$$

- *Dual problem:*
  Find 'best' (α,β,θ) s.t
  $L(x,y) = |y-x| - \alpha(y) - \beta(x) + \theta(x)(y-x) \geq 0$ for all (x,y).

- Approach à la Gangbo/McCann (concave cost):
  i.) Don't move common mass, ii.) look for a decr. monotone split.

Martingale splitting



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$$L(x,x) = L(x,p(x)) = L(x,q(x)) = 0$$

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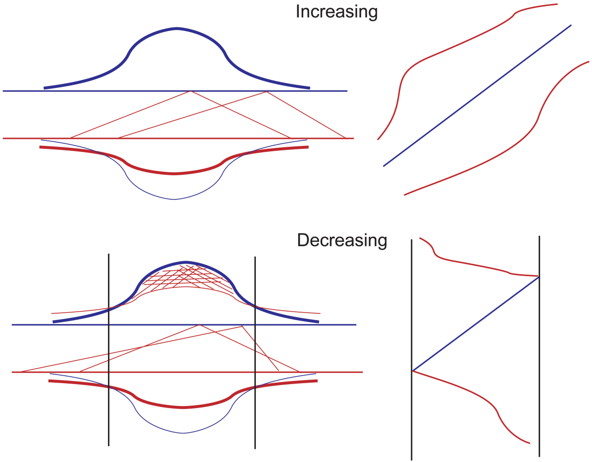
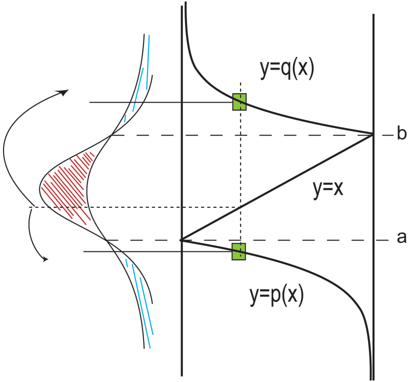


Map $\eta = (\mu - \nu)^+$ to $\gamma = (\nu - \mu)^+$



$$\int_a^z f_\eta(u)du = \int_{q(z)}^{\infty} f_\gamma(u)du + \int_{p(z)}^{a} f_\gamma(u)du$$

$$\int_a^z uf_\eta(u)du = \int_{q(z)}^{\infty} uf_\gamma(u)du + \int_{p(z)}^{a} uf_\gamma(u)du$$

Differential equations

Map $\eta = (\mu - \nu)^+$ to $\gamma = (\nu - \mu)^+$ with constraints

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where $p : [a, b] \to (-\infty, a]$, $q : [a, b] \to [b, \infty)$. *Differentiate to obtain a pair of differential equations:*

$$\begin{aligned} p'(x) &= \frac{q(x) - x}{q(x) - p(x)} \frac{f_\mu(x) - f_\nu(x)}{f_\mu(p(x)) - f_\nu(p(x))}, \\ q'(x) &= \frac{x - p(x)}{q(x) - p(x)} \frac{f_\mu(x) - f_\nu(x)}{f_\mu(q(x)) - f_\nu(q(x))}. \end{aligned}$$

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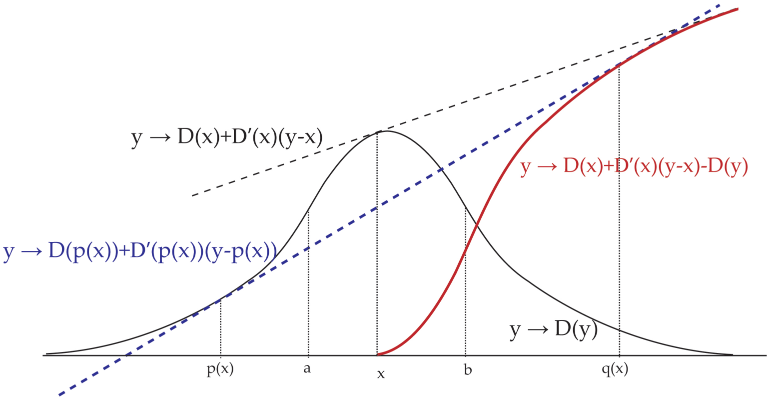
\end{aligned}$$

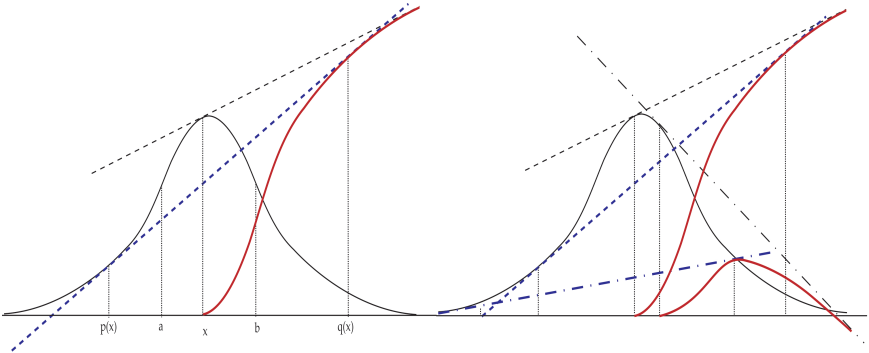
Better: A 'potential' picture

Let $D(x) = \int (y-x)^+ \nu(dy) - \int (y-x)^+ \mu(dy)$ Manipulating the two equations a bit, find that (p, q) is given by

$$\begin{aligned} D'(x) &= D'(p(x)) + D'(q(x)) \\ D(x) - xD'(x) &= D(q(x)) - q(x)D'(q(x)) + D(p(x)) - p(x)D'(p(x)) \end{aligned}$$



Monotonicity in the potential picture



The optimal coupling

Theorem (DGH&K '13)
There exists a unique pair of decreasing functions (p, q) such that if $X \sim \mu$ and $Y \in \{p(X), X, q(X)\}$ with $\mathbb{E}[Y|X] = X$ then $Y \sim \nu$ and (X, Y) minimises $\mathbb{E}[|Y - X|]$ over all martingale couplings.
The joint measure is

$$\rho(x,y) = \begin{cases} f_\eta(x)\frac{q(x)-x}{q(x)-p(x)} I_{\{y=p(x)\}} & y < x, \\ f_\mu(x) - f_\eta(x) & y = x, \\ f_\eta(x)\frac{x-p(x)}{q(x)-p(x)} I_{\{y=q(x)\}} & y > x. \end{cases}$$

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There exists a unique pair of decreasing functions (p, q) such that if $X \sim \mu$ and $Y \in \{p(X), X, q(X)\}$ with $\mathbb{E}[Y|X] = X$ then $Y \sim \nu$ and (X, Y) minimises $\mathbb{E}[|Y - X|]$ over all martingale couplings.
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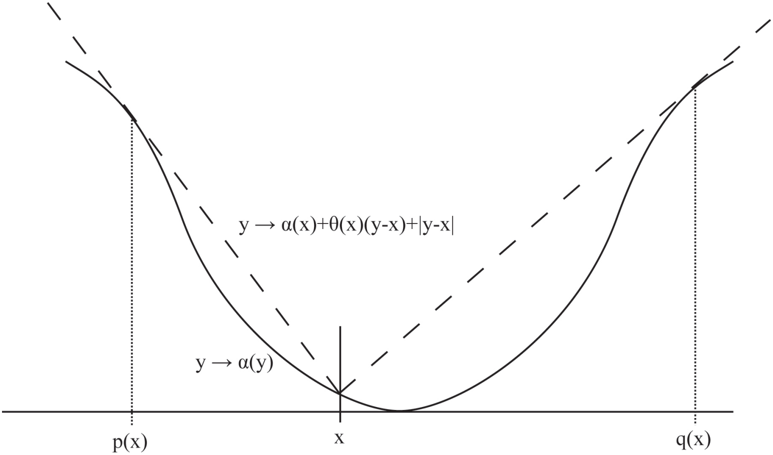
$$\rho(x,y) = \begin{cases} f_\eta(x)\frac{q(x)-x}{q(x)-p(x)} I_{\{y=p(x)\}} & y < x, \\ f_\mu(x) - f_\eta(x) & y = x, \\ f_\eta(x)\frac{x-p(x)}{q(x)-p(x)} I_{\{y=q(x)\}} & y > x. \end{cases}$$

The dual picture



$$L(x,y) = |y-x| - \alpha(y) + \alpha(x) - \theta(x)(x-y)$$
$$L(x,x) = L(x,p(x)) = L_y(x,p(x)) = L(x,q(x)) = L_y(x,q(x)) = 0$$

Expanding, for $x \in [a, b]$,

$$\begin{aligned} x - p(x) + \alpha(x) - \alpha(p(x)) - \theta(x)(x - p(x)) &= 0 \quad (1) \\ -1 - \alpha'(p(x)) + \theta(x) &= 0 \quad (2) \\ q(x) - x + \alpha(x) - \alpha(q(x)) - \theta(x)(x - q(x)) &= 0 \quad (3) \\ 1 - \alpha'(q(x)) + \theta(x) &= 0 \quad (4) \end{aligned}$$

Differentiating (1) and using (2),

$$1 + \alpha'(x) - \theta(x) - \theta'(x)(x - p(x)) = 0. \quad (5)$$

Similarly, $-1 + \alpha'(x) - \theta(x) - \theta'(x)(x - q(x)) = 0$ (6).
Subtracting (6) from (5) gives

$$\theta'(x) = \frac{2}{q(x) - p(x)}.$$

Adding (5) and (6),

$$\alpha'(x) = \theta(x) + \frac{\theta'(x)}{2}(2x - p(x) - q(x)).$$

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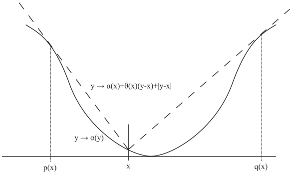
Properties

$$L(x,y) = |y-x| + \alpha(x) - \alpha(y) - \theta(x)(x-y).$$



- $q(x)$ is a local minimum in y for L(x,y), expect $0 \leq L_{yy}(x,q(x)) = -\alpha''(q(x))$. So α is concave at $y = q(x)$.
- Also $\alpha'(q(x)) = -1 + \theta(x)$, so $\alpha''(q(x))q'(x) = \theta'(x) > 0$. So $q'(x) < 0$.

The optimal sub-hedge

Fix $x_0 \in [a, b]$, define $\theta = [a, b] \to \mathbb{R}$ and $\alpha = [a, b] \to \mathbb{R}$ via

$$\begin{aligned} \theta(x) &= \int_{x_0}^{x} \frac{2dz}{q(z) - p(z)}, \\ \alpha(x) &= \int_{x_0}^{x} \frac{2x - q(z) - p(z)}{q(z) - p(z)} dz = x\theta(x) - \int_{x_0}^{x} \frac{q(z) + p(z)}{q(z) - p(z)} dz. \end{aligned}$$

Extend to $\mathbb{R}$ by defining $\delta : \mathbb{R} \to \mathbb{R}$ and $\psi : [a, b] \to \mathbb{R}$ via

$$\delta(x) = \begin{cases} \theta(p^{-1}(x)) & x < a, \\ \theta(x) & x \in [a, b], \\ \theta(q^{-1}(x)) & x > b. \end{cases}$$

$$\psi(x) = \begin{cases} \alpha(p^{-1}(x)) + (p^{-1}(x) - x)(1 - \theta(p^{-1}(x))) & x < a, \\ \alpha(x) & x \in [a, b], \\ \alpha(q^{-1}(x)) + (q^{-1}(x) - x)(-1 - \theta(q^{-1}(x))) & x > b. \end{cases}$$

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\end{aligned}
$$

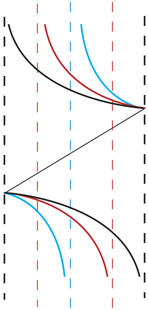
The optimal sub-hedge

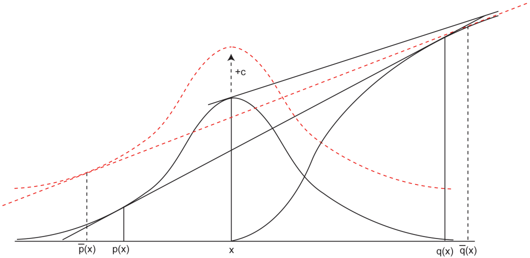
Theorem (DGH&K '13)
$L(x,y) = |y-x| + \psi(x) - \psi(y) - \delta(x)(x-y) \geq 0$ for all $x, y \in \mathbb{R}$,
with equality for $y \in \{p(x), x, q(x)\}$.

Monge push-forward constraint ∼ martingale constraint

The martingale constraint is comparable to the Monge constraint

Consider a sequence of relaxed problems?





Uniform example

Suppose $\mu \sim U[-1,1]$ and $\nu \sim U[-2,2]$.

$$D(x) = \begin{cases} \frac{1}{8}x^2 + \frac{1}{2}x + \frac{1}{2} & -2 \le x \le -1, \\ \frac{1}{4} - \frac{1}{8}x^2 & -1 < x < 1, \\ \frac{1}{8}x^2 - \frac{1}{2}x + \frac{1}{2} & 1 \le x \le 2. \end{cases}$$

For $-1 < x < 1$ we have

$$\begin{aligned} D'(x) &= D'(p(x)) + D'(q(x)) \Rightarrow -x = q(x) + p(x) \\ D(x) - xD'(x) &= D(q(x)) - q(x)D'(q(x)) + D(p(x)) - p(x)D'(p(x)) \\ &= D(q(x)) - q(x)D'(q(x)) + D(-x-q(x)) \\ &+ (x+q(x))D'(-x-q(x)). \end{aligned}$$

After some simplification we find $q(x)^2 + q(x)x + x^2 - 3 = 0$, so $q(x) = \frac{-x+\sqrt{12-3x^2}}{2}$, and $p(x) = \frac{-x-\sqrt{12-3x^2}}{2}$.

Uniform example: Dual

Setting $x_0 = 0$, we have

$$\theta(x) = \int_0^x \frac{2}{\sqrt{12 - 3u^2}} du = \frac{1}{\sqrt{3}} \int_0^x \frac{du}{\sqrt{1 - u^2/4}} = \frac{2}{\sqrt{3}} \sin^{-1}\left(\frac{x}{2}\right)$$

and

$$\alpha(x) = x\theta(x) + \int_0^x \frac{u du}{\sqrt{12 - 3u^2}} = \frac{2x}{\sqrt{3}} \sin^{-1}\left(\frac{x}{2}\right) + \frac{2 - \sqrt{4 - x^2}}{\sqrt{3}}.$$

Recap

- Common theme in (m-g) optimal coupling: Monotonicity stemming from a 'supermodular structure'
- There is a natural link to a literature on mechanism design
- The mechanism design view is not absurd in a finance/ 'martingale setting'
- Intuition for Monge's cost function;
- Solved using the Lagrangian approach
- Feature: 'Double gradient potential picture'
- Martingale constraint looks a bit like the Monge constraint...

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Some references

- Étienne Ghys, Gaspard Monge: Le mémoire sur les déblais et les remblais, http://images.math.cnrs.fr/Gaspard-Monge,1094.html
- Rockafellar, Knott-Smith, Rüschendorf, Brenier.
  The 'gradient principle'
- Rüschendorf '07, 'Monge-Kantorovich transportation problem and optimal couplings'
- Beiglböck/Juillet '12 (Touzi/Labordère '13), Hobson/Neuberger '08 (Hobson/K '13), Duembgen/Rogers (advertised) http://www.mathnet.ru/php/presentation.phtml?option_lang=eng&presentid=5314.
- 'Mechanism design' view, etc: Hobson/K '12...